

A fixed point theorem for contractive mappings that characterizes metric completeness

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Abstract

Inspired by the work of Suzuki in [Proc. Amer. Math. Soc. **136** (2008), 1861–1869] we prove a fixed point theorem for contractive mappings that generalizes a theorem of Geraghty in [Proc. Amer. Math. Soc., **40** (1973), 604–608] and characterizes metric completeness.¹

1 Introduction

Throughout this paper, we denote by \mathbb{N} the set of all positive integers, by \mathbb{Z}^+ the set of nonnegative integers, and by \mathbb{R}^+ the set of nonnegative real numbers. Given a set X and a mapping $T : X \rightarrow X$, the n th iterate of T is denoted by T^n so that $T^2x = T(Tx)$, $T^3x = T(T^2x)$ and so on. A point $x \in X$ is called a *fixed point* of T if $T(x) = x$.

Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is called a *contraction* if there is $r \in [0, 1)$ such that

$$\forall x, y \in X, \quad d(Tx, Ty) \leq rd(x, y). \quad (1)$$

The following famous theorem is referred to as the Banach contraction principle.

Theorem 1 (Banach, [1]). *If (X, d) is a complete metric space, then every contraction T on X has a unique fixed point.*

The Banach fixed point theorem is very simple and powerful. It became a classical tool in nonlinear analysis with many generalizations; see [2, 3, 4, 7, 12, 13, 14, 19, 20, 21, 22, 23, 24]. For instance, the following result due to Boyd and Wong is a great generalization of Theorem 1.

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Theorem 2 (Boyd and Wong, [2]). *Let (X, d) be a complete metric space, and let T be a mapping on X . Assume there exists a right-continuous function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\varphi(s) < s$ for $s > 0$, and*

$$\forall x, y \in X, \quad d(Tx, Ty) \leq \varphi(d(x, y)). \quad (2)$$

Then T has a unique fixed point.

There is an example of an incomplete metric space X on which every contraction has a fixed point, [5]. This means that Theorem 1 cannot characterize the metric completeness of X . Recently, Suzuki in [23] proved the following remarkable generalization of the classical Banach contraction theorem that characterizes the metric completeness of X .

Theorem 3 (Suzuki, [23]). *Define a function $\theta : [0, 1) \rightarrow (1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq (\sqrt{5} - 1)/2; \\ (1 - r)r^{-2}, & \text{if } (\sqrt{5} - 1)/2 \leq r \leq 1/\sqrt{2}; \\ (1 + r)^{-1}, & \text{if } 1/\sqrt{2} \leq r < 1. \end{cases}$$

Let (X, d) be a metric space. Then X is complete if and only if every mapping T on X satisfying the following has a fixed point:

- *There exists $r \in [0, 1)$ such that*

$$\forall x, y \in X \quad (\theta(r)d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq rd(x, y)). \quad (3)$$

The above Suzuki's generalized version of Banach fixed point theorem initiated a lot of work in this direction and led to some important contribution in metric fixed point theory. Several authors obtained variations and refinements of Suzuki's result; see [8, 10, 11, 15, 17, 18].

For a metric space (X, d) , a mapping $T : X \rightarrow X$ is called *contractive* if $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$ with $x \neq y$. Edelstein in [6] proved that, on compact metric spaces, every contractive mapping possesses a unique fixed point theorem. Then in [24] Suzuki generalized Edelstein's result as follows.

Theorem 4 (Suzuki, [24]). *Let (X, d) be a compact metric space and let T be a mapping on X . Assume that*

$$\forall x, y \in X \quad \left(\frac{1}{2}d(x, Tx) < d(x, y) \implies d(Tx, Ty) < d(x, y) \right). \quad (4)$$

Then T has a unique fixed point.

It is interesting to note that, although the above Suzuki's theorem generalizes Edelstein's theorem in [6], these two theorems, as Suzuki mentioned in [24], are not of the same type.

Let T be a contractive mapping on a metric space X . Choose a point $x \in X$ and set $x_n = T^n x$, for $n \in \mathbb{N}$. Criteria for the sequence of iterates $\{x_n\}$ to

be Cauchy are of interest, for if it is Cauchy then it converges to a unique fixed point of T , [9]. Many papers have presented such criteria, especially since the important paper of Rakotch [16]. For example, Geraghty in [9] proved the following theorem that gives a necessary and sufficient condition for a sequence of iterates to be convergent.

Theorem 5 (Geraghty, [9]). *Let X be a complete metric space and let T be a contractive mapping on X . Let $x \in X$ and set $x_n = T^n x$, $n \in \mathbb{N}$. Then x_n converges to a unique fixed point of T if and only if for any two subsequences $\{x_{p_n}\}$ and $\{x_{q_n}\}$, with $x_{p_n} \neq x_{q_n}$, if $\Delta_n \rightarrow 1$ then $\delta_n \rightarrow 0$, where*

$$\delta_n = d(x_{p_n}, x_{q_n}), \quad \Delta_n = d(Tx_{p_n}, Tx_{q_n})/\delta_n.$$

Motivated by the works of Suzuki in [23] and [24], we prove a fixed point theorem for contractive mappings based on Theorem 5 that characterizes metric completeness.

2 Fixed Point Theorem

Let (X, d) be a metric space. We shall use the following notation: for any pair of subsequences $\{x_{p_n}\}$ and $\{x_{q_n}\}$ of a given sequence $\{x_n\}$ in X , we let $\delta_n = d(x_{p_n}, x_{q_n})$ and

$$\Delta_n = \begin{cases} 0, & \delta_n = 0; \\ d(Tx_{p_n}, Tx_{q_n})/\delta_n, & \delta_n > 0. \end{cases}$$

Theorem 6. *Let (X, d) be a metric space and let a mapping $T : X \rightarrow X$ satisfy the following condition:*

$$\forall x, y \in X \left(x \neq y, d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) < d(x, y) \right). \quad (5)$$

Given $x \in X$, the following statements for the sequence $x_n = T^n x$, $n \in \mathbb{N}$, are equivalent:

- (i) $\{x_n\}$ is a Cauchy sequence.
- (ii) For any two subsequences $\{x_{p_n}\}$ and $\{x_{q_n}\}$, with $d(x_{p_n}, Tx_{p_n}) \leq d(x_{p_n}, x_{q_n})$ for all n , if $\Delta_n \rightarrow 1$ then $\delta_n \rightarrow 0$.

Proof. The implication (i) \Rightarrow (ii) is clear because $\{x_n\}$ is a Cauchy sequence and thus for any two subsequences $\{x_{p_n}\}$ and $\{x_{q_n}\}$ we always have $\delta_n \rightarrow 0$.

We now prove (ii) \Rightarrow (i). First, assume that $x_m = Tx_m$, for some m . Then $x_n = x_m$, for $n \geq m$, and particularly $\{x_n\}$ is a Cauchy sequence. Next, assume that $x_n \neq x_{n+1}$ for all n . Since $d(x_n, Tx_n) \leq d(x_n, Tx_n)$, condition (5) implies that the sequence $\delta_n = d(x_n, x_{n+1})$ is strictly decreasing. Thus $\delta_n \rightarrow \delta$ for some nonnegative number δ . If $\delta > 0$, take $p_n = n$ and $q_n = n + 1$. Then $d(x_{p_n}, Tx_{p_n}) \leq d(x_{p_n}, x_{q_n})$, for all n , and $\Delta_n \rightarrow 1$ while $\delta_n \rightarrow \delta \neq 0$. This is a contradiction and hence $d(x_n, x_{n+1}) \rightarrow 0$.

For every $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ such that $d(x_m, x_{m+1}) < 1/n$ for $m \geq k_n$. If $\{x_n\}$ is not a Cauchy sequence, there exist $\varepsilon > 0$ and sequences $\{p_n\}$ and $\{q_n\}$ of positive integers such that $q_n > p_n \geq k_n$ and $d(x_{p_n}, x_{q_n}) \geq \varepsilon$. We also assume that q_n is the least such integer so that $d(x_{p_n}, x_{q_n-1}) < \varepsilon$. Therefore,

$$\varepsilon \leq d(x_{p_n}, x_{q_n}) \leq d(x_{p_n}, x_{q_n-1}) + d(x_{q_n-1}, x_{q_n}) < \varepsilon + 1/n.$$

This shows that $\delta_n \rightarrow \varepsilon$. Since we have, for every $n \in \mathbb{N}$,

$$d(x_{p_n}, Tx_{p_n}) \leq d(x_{p_n}, x_{q_n}) < d(x_{p_n}, x_{q_n}),$$

condition (5) shows that $d(Tx_{p_n}, Tx_{q_n}) < \delta_n$. So

$$\frac{\delta_n - 2/n}{\delta_n} \leq \frac{d(Tx_{p_n}, Tx_{q_n})}{\delta_n} = \Delta_n < 1.$$

It shows that $\Delta_n \rightarrow 1$ and thus $\delta_n \rightarrow 0$. This is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. \square

The following is a Susuki-type generalization of Theorem 5.

Theorem 7. *Let X be a complete metric space and let T be a mapping on X satisfying the following condition:*

$$\forall x, y \in X \left(\frac{1}{2}d(x, Tx) < d(x, y) \implies d(Tx, Ty) < d(x, y) \right). \quad (6)$$

Given $x \in X$, the following statements for the sequence $x_n = T^n x$, $n \in \mathbb{N}$, are equivalent:

- (i) $x_n \rightarrow z$ in X , with z a unique fixed point of T ;
- (ii) for any two subsequences $\{x_{p_n}\}$ and $\{x_{q_n}\}$, with $d(x_{p_n}, Tx_{p_n}) \leq d(x_{p_n}, x_{q_n})$ for all n , if $\Delta_n \rightarrow 1$ then $\delta_n \rightarrow 0$.

Proof. First, let us prove that T has at most one fixed point. If z is a fixed point of T and $z \neq y$ then $(1/2)d(z, Tz) < d(z, y)$ and condition (6) implies that $d(Tz, Ty) < d(z, y)$. Since $Tz = z$, we must have $Ty \neq y$, i.e., y is not a fixed point of T .

The implication (i) \Rightarrow (ii) is clear. We prove (ii) \Rightarrow (i). By Theorem 5, the sequence $\{x_n\}$ is Cauchy and, since the metric space X is complete, $x_n \rightarrow z$ for some $z \in X$. We show that $Tz = z$. First note that,

$$\forall n \left(d(x_n, x_{n+1}) < 2d(x_n, z) \quad \text{or} \quad d(x_{n+1}, x_{n+2}) < 2d(x_{n+1}, z) \right). \quad (7)$$

In fact, if $2d(x_n, z) \leq d(x_n, x_{n+1})$ and $2d(x_{n+1}, z) \leq d(x_{n+1}, x_{n+2})$ hold, for some n , then

$$\begin{aligned} 2d(x_n, x_{n+1}) &\leq 2d(x_n, z) + 2d(x_{n+1}, z) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &< d(x_n, x_{n+1}) + d(x_n, x_{n+1}) = 2d(x_n, x_{n+1}). \end{aligned}$$

This is absurd and thus (7) must hold. Now condition (6) together with (7) imply that

$$\forall n \left(d(x_{n+1}, Tz) < d(x_n, z) \quad \text{or} \quad d(x_{n+2}, Tz) < d(x_{n+1}, z) \right). \quad (8)$$

Since $x_n \rightarrow z$, condition (8) implies the existence of a subsequence of $\{x_n\}$ that converges to Tz . This shows that $Tz = z$. \square

Remark. In Theorem 7, if we replace condition (6) with condition (9) below, we are still able to prove the theorem except for the uniqueness of the fixed point;

$$\forall x, y \in X \left(\frac{1}{2}d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq d(x, y) \right), \quad (9)$$

We next prove that $1/2$ in condition (6) of Theorem 7 is the best constant.

Theorem 8. *For every $\eta \in (1/2, \infty)$, there exist a complete metric space (X, d) and a mapping $T : X \rightarrow X$ with the following properties:*

- (i) *the mapping T has no fixed point in X ,*
- (ii) *$\eta d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$,*
- (iii) *condition (ii) of Theorem 7 holds for any choice of initial point.*

Proof. We use the same method as in [24, Theorem 4]. Take $\eta \in (1/2, \infty)$ and choose $r \in (1/\sqrt{2}, 1)$ such that $(1+r)^{-1} < \eta$. For $n \in \mathbb{Z}^+$, let $u_n = (1-r)(-r)^n$, and then set $X = \{0, 1\} \cup \{u_n : n \in \mathbb{Z}^+\}$. Define a mapping T on X by $T0 = 1$, $T1 = u_0$ and $Tu_n = u_{n+1}$ for $n \in \mathbb{Z}^+$. Obviously T has no fixed point in X and thus (i) is proved. We now prove part (ii). In [23], Suzuki showed the following

$$\forall x, y \in X \left((1+r)^{-1}d(x, Tx) < d(x, y) \implies d(Tx, Ty) \leq rd(x, y) \right).$$

Now, if $\eta d(x, Tx) \leq d(x, y)$ then $(1+r)^{-1}d(x, Tx) < d(x, y)$ and thus $d(Tx, Ty) \leq rd(x, y) < d(x, y)$. This proves part (ii). Finally, we show that, in this setting, condition (ii) of Theorem 7 holds. Take an arbitrary element $x \in X$ as initial point and set $x_n = T^n x$, $n \in \mathbb{N}$. Then $\{x_n : n \geq 2\}$ is a subsequence of $\{u_n\}$ and since $u_n \rightarrow 0$ the sequence $\{x_n\}$ is Cauchy. Hence if $\{x_{p_n}\}$ and $\{x_{q_n}\}$ are two subsequences of $\{x_n\}$ we have $d(x_{p_n}, x_{q_n}) \rightarrow 0$. \square

Using a similar method as in [9], we can easily convert the sequential condition (ii) in Theorem 7 to the more customary functional form. Following [9], we define a class of test functions as follows:

Definiton 9. We say that ψ is of class Ψ , written $\psi \in \Psi$, if ψ is a function of \mathbb{R}^+ into $[0, 1]$ and, for every sequence $\{s_n\}$ of positive numbers, the condition $\psi(s_n) \rightarrow 1$ implies that $s_n \rightarrow 0$.

We do not assume that ψ is continuous in any sense. We only require that if ψ gets near one, it does so only near zero, [9].

Theorem 10. *Let X be a complete metric space and let T be a mapping on X satisfying (6). For any $x \in X$, the following statements, for the sequence $x_n = T^n x$, $n \in \mathbb{N}$, are equivalent:*

- (i) $x_n \rightarrow z$ in X , with z the unique fixed point of T ;
- (ii) there is $\psi \in \Psi$ such that, for all $m, n \in \mathbb{N}$,

$$d(x_n, Tx_n) \leq d(x_n, x_m) \implies d(Tx_n, Tx_m) \leq \psi(d(x_n, x_m))d(x_n, x_m). \quad (10)$$

Proof. It suffices to show that condition (ii) of the theorem is equivalent to condition (i) of Theorem 7. First assume that $\psi \in \Psi$ exists and satisfies (10). Let $\{x_{p_n}\}$ and $\{x_{q_n}\}$ be subsequences of $\{x_n\}$ such that $d(x_{p_n}, Tx_{p_n}) \leq d(x_{p_n}, x_{q_n})$. Assume that $\Delta_n \rightarrow 1$. Since $d(x_{p_n}, Tx_{p_n}) \leq d(x_{p_n}, x_{q_n})$, condition (10) shows that $\psi(\delta_n) \rightarrow 1$. Since $\psi \in \Psi$, we have $\delta_n \rightarrow 0$.

Next assume that the sequential condition (ii) of Theorem (7) holds. Define $\psi : \mathbb{R}^+ \rightarrow [0, 1]$ as follows: Given $s \in \mathbb{R}^+$, if there exist no $m, n \in \mathbb{N}$ for which $d(x_n, Tx_n) \leq s \leq d(x_n, x_m)$, define $\psi(s) = 0$; otherwise define

$$\psi(s) = \sup \left\{ \frac{d(Tx_n, Tx_m)}{d(x_n, x_m)} : d(x_n, Tx_n) \leq s \leq d(x_n, x_m) \right\}.$$

Since T satisfies condition (6), we have $0 \leq \psi(s) \leq 1$, for every s . Assume that $\psi(s_n) \rightarrow 1$ for some sequence $\{s_n\}$ in \mathbb{R}^+ . Take a sequence $\{r_n\}$ of positive numbers such that $r_n < \psi(s_n)$ and $r_n \rightarrow 1$. Then, there exist two subsequences $\{x_{p_n}\}$ and $\{x_{q_n}\}$ for which $d(x_{p_n}, Tx_{p_n}) \leq s_n \leq d(x_{p_n}, x_{q_n})$ and

$$r_n < \frac{d(Tx_{p_n}, Tx_{q_n})}{d(x_{p_n}, x_{q_n})} \leq \psi(s_n).$$

Therefore, $\Delta_n \rightarrow 1$ and condition (ii) of Theorem 7 shows $\delta_n \rightarrow 0$. \square

We now apply the above results to obtain a criterion for convergence of the iteration from an arbitrary starting point.

Theorem 11. *Let T be a mapping on a complete metric space X . Assume that, for some $\psi \in \Psi$, we have*

$$\forall x, y \in X \left(\frac{1}{2}d(x, Tx) < d(x, y) \implies d(Tx, Ty) < \psi(d(x, y))d(x, y) \right). \quad (11)$$

Then for any choice of initial point x , the iteration $x_n = T^n x$, $n \in \mathbb{N}$, converges to the unique fixed point z of T in X .

3 Metric Completion

In this section, we discuss the metric completeness.

Theorem 12. *Let (X, d) be a metric space. Then X is complete if and only if every mapping $T : X \rightarrow X$ satisfying the following two conditions has a fixed point in X ;*

- (i) *There exists a constant $\eta \in (0, 1/2]$ such that $\eta d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$.*
- (ii) *There exists a point $x \in X$ such that condition (ii) of Theorem 7 holds.*

Theorem 13. *For a metric space (X, d) , the following are equivalent:*

- (i) *The space X is complete.*
- (ii) *For any mapping T on X that satisfies (6), conditions (i) and (ii) of Theorem 7 are equivalent.*

Proof. (i) \Rightarrow (ii) follows from Theorem 7. To prove (ii) \Rightarrow (i), towards a contradiction, let the metric space X be incomplete. Then, as in the proof of Theorem 4 in [23], there exists a Cauchy sequence $\{u_n\}$ which does not converge. Define a function $\rho : X \rightarrow \mathbb{R}^+$ by $\rho(x) = \lim_n d(x, u_n)$, for $x \in X$. Note that ρ is well-defined because $\{d(x, u_n)\}$ is a Cauchy sequence in \mathbb{R} , for every $x \in X$. The following are obvious:

- $\rho(x) - \rho(y) \leq d(x, y) \leq \rho(x) + \rho(y)$, for $x, y \in X$,
- $\rho(x) > 0$ for all $x \in X$,
- $\rho(u_n) \rightarrow 0$ as $n \rightarrow \infty$.

Define a mapping $T : X \rightarrow X$ as follows: For each $x \in X$, since $\rho(x) > 0$ and $\rho(u_n) \rightarrow 0$, there exists $m \in \mathbb{N}$ such that

$$\rho(u_n) < \frac{\rho(x)}{7}, \quad (n \geq m). \quad (12)$$

Put $T(x) = u_m$. In case $x = u_k$, for some k , we choose m large enough such that $m > k$ and (12) holds. It is obvious that $\rho(Tx) < \rho(x)/7$ so that $Tx \neq x$, for every $x \in X$. That is, T does not have a fixed point. Let us prove that T satisfies (6). Fix $x, y \in X$ with $(1/2)d(x, Tx) < d(x, y)$. In the case where $2\rho(x) \leq \rho(y)$, we have

$$\begin{aligned} d(Tx, Ty) &\leq \rho(Tx) + \rho(Ty) < \frac{1}{3}(\rho(x) + \rho(y)) \\ &\leq \frac{1}{3}(\rho(x) + \rho(y)) + \frac{2}{3}(\rho(y) - 2\rho(x)) \\ &= \rho(y) - \rho(x) \leq d(x, y). \end{aligned}$$

In the other case, where $\rho(y) < 2\rho(x)$, we have

$$d(x, y) > \frac{1}{2}d(x, Tx) \geq \frac{1}{2}(\rho(x) - \rho(Tx)) \geq \frac{1}{2}\left(1 - \frac{1}{7}\right)\rho(x) = \frac{3}{7}\rho(x).$$

Therefore,

$$\begin{aligned} d(Tx, Ty) &\leq \rho(Tx) + \rho(Ty) < \frac{1}{7}(\rho(x) + \rho(y)) \\ &\leq \frac{1}{7}(\rho(x) + 2\rho(x)) = \frac{3}{7}\rho(x) \leq d(x, y). \end{aligned}$$

Finally, we show that, given $x \in X$, condition (ii) of Theorem 7 holds for the iteration sequence $x_n = T^n x$, $n \in \mathbb{N}$. The definition of T shows that there exists a sequence $\{m_n\}$ of positive integers such that $m_n < m_{n+1}$ and $x_n = u_{m_n}$. Hence $\{x_n\}$ is a subsequence of $\{u_n\}$. Now, if $\{x_{p_n}\}$ and $\{x_{q_n}\}$ are subsequences of $\{x_n\}$, they are also subsequences of $\{u_n\}$ and thus $d(x_{p_n}, x_{q_n}) \rightarrow 0$ because $\{u_n\}$ is a Cauchy sequence. This shows that condition (ii) of Theorem 7 holds for the sequence $\{x_n\}$. This is a contradiction since condition (i) of Theorem 7 does not hold for the sequence $\{x_n\}$. \square

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